SPECTRAL RESIDUES OF SECOND-ORDER DIFFERENTIAL EQUATIONS: A NEW METHODOLOGY FOR SUMMATION IDENTITIES AND INVERSION FORMULAS.

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Abstract. The present article deals with differential equations with spectral parameter from the point of view of formal power series. The treatment does not make use of the notion of eigenvalue, but introduces a new idea: the spectral residue.

The article focuses on second-order, self-adjoint problems. In such a setting every potential function determines a sequence of spectral residues. This correspondence is invertible, and gives rise to a combinatorial inversion formula. Other interesting combinatorial consequences are obtained by considering spectral residues of exactly-solvable potentials of 1-dimensional quantum mechanics.

It is also shown that the Darboux transformation of 1-dimensional potentials corresponds to a simple negation of the corresponding spectral residues. This fact leads to another combinatorial inversion formula.

Finally, there is a brief discussion of applications. The topics considered are enumeration problems and integrable systems.

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1. Introduction

There is a rich interplay between the methods and ideas of theoretical physics and that of combinatorics. The connection runs both ways: combinatorial methods are applied to the study of physical models, while ideas from physics are used to obtain results of a purely combinatorial nature.

The present paper follows the second pattern. The aim is to demonstrate that certain methods for the solution of the time-independent Scrhödinger equation lead naturally to interesting identities and inversion formulas. The idea that allows us to go from the one domain to the other is that of a spectral residue. The application of differential operators and residue techniques to combinatorial identities and inversion formulas is well documented [1, 2, 3, 4]. However, the notion of a spectral residue seems to be unknown, and therefore deserves to be considered in some detail.

The impetus for spectral residues comes from the following problem: to give a formal series treatment of differential equations with spectral parameter. It is far from clear whether the notion of eigenvalue is meaningful in a formal setting, and so we take a different approach. Consider a linear differential operator T in one independent variable z, with spectral parameter ν, and define $\phi(z;\nu)$ as the formal series solution of $T[\phi] = 0$. As the analogue of the spectrum of T, let us take the set of values of ν for which some of the coefficients of ϕ are singular. This spectral set is not of interest in itself; in the cases considered here it is just the set of natural numbers. More interesting are the residue of the coefficients of the solution at the singular values of ν; these we will call the spectral residues of the operator.

Let us restrict our attention to classical quantum-mechanics and consider second-order, differential equations that are amenable to the usual method of undetermined coefficients (aka the method of Frobenius):

$$T[\phi] = z^2\phi''(z) + (1 - \nu + P(z))z\phi'(z) + Q(z)\phi(z) = 0,\ \phi(0) = 1.$$

Here P and Q are formal power series in z with vanishing constant term, and ν is the spectral parameter. The roots of the indicial equation are 0 and ν. Hence if ν is a natural number, then, in general, the solution is singular. In this way there arises a sequence of spectral residues, one for each natural number. If we can solve the operator equation $T[\phi] = 0$, then we will be able to obtain information about this sequence.

From the very beginning of quantum mechanics, the study of Schrödinger's equation was marked by the search for exact solutions. The two outstanding methods in this endeavor are first, the theory of hypergeometric functions[5], and second the Darboux transformation[6, 7].

The first method furnishes a large class of exactly-solvable potentials whose eigenfunctions can be given in terms of either the hypergeometric or the confluent hypergeometric functions[8]. All the classical exactly-solvable potentials: the Harmonic oscillator, the Coulomb, the Morse, the Eckart potentials, are of this type.

As for the second method, the essential feature of the Darboux transformation is that it changes one potential function into another in such a way that the eigenfunctions of the one can be explicitly constructed from the eigenfunctions of the other. By means of this pairing exactly solvable potentials yield new exactly solvable potentials. Combining the two methods it becomes possible to describe a very large class of exactly solvable potentials.

Our goal is to examine these two methodologies from a combinatorial standpoint by using spectral residues. In this context the following questions are quite natural.

1. Does the spectral residue sequence completely determine the potential? If yes, can this relationship be described explicitly?
2. What are the spectral residue sequences corresponding to potentials of hypergeometric type? Does the exact-solvability of these potentials manifest itself at the level of spectral residues?
3. What is the effect of the Darboux transformation on the corresponding spectral residue sequence?

We will see that all of these questions have satisfactory answers, and that these answers manifest themselves as a novel class of summation identities and inversion formulas. The sums that occur in this context are indexed by compositions, i.e. ordered partitions. More specifically, we will say that an l-part composition of a natural number n is an *ordered* list of l positive integers $\mathbf{p} = (p_1, p_2, \dots, p_l)$ such that

$$p_1 + \ldots + p_l = n.$$

On occasion we will write $|\mathbf{p}|$ instead of n, and $\ell(\mathbf{p})$ instead of l.

Identities that feature sums indexed by partitions are not new [9][10, e.g. §60][11]. All of these references feature sums whose terms are symmetric functions of the partition entries. Thus, no generality is lost by working with partitions rather than compositions. By contrast, the sums that arise in connection with spectral residues have terms that are functions of the partial sums of the composition entries; the order of the entries cannot be ignored.

For a composition $\mathbf{p} = (p_1, \dots, p_l)$ we define

$$s_{\mathbf{p}} = s_1 s_2 \dots s_l,$$

where $s_j = p_1 + \ldots + p_j$ is the j^{th} left partial sum corresponding to $\mathbf{p}$. Thus, if one interprets $\mathbf{p}$ as a walk from 0 to n with steps $p_1, \ldots, p_l$, then $s_{\mathbf{p}}$ is the product of numbers visited on that walk. Let $\mathbf{p}'$ denote the reversed composition $(p_l, \ldots, p_2, p_1)$. In this way $s_{\mathbf{p}'}$ is equal to the product of visited numbers when one interprets $\mathbf{p}$ as a reverse walk from n down to 0.

Let $U_1, U_2, U_3, \ldots$ be an infinite sequence of non-commuting variables. For $n = 1, 2, \ldots$ set

$$\rho_n = \sum_{|\mathbf{p}|=n} \frac{n}{s_{\mathbf{p}}\, s_{\mathbf{p}'}}\, U_{\mathbf{p}} \tag{1.1}$$

where the index runs over all compositions $\mathbf{p} = (p_1, \ldots, p_l)$ of n, and $U_{\mathbf{p}}$ is an abbreviation for the l-fold product $U_{p_1} U_{p_2} \ldots U_{p_l}$. Here are the first few elements of the resulting sequence:

$$\begin{aligned}
\rho_1 &= U_1, \\
\rho_2 &= \frac{1}{2} U_2 + \frac{1}{2} {U_1}^2, \\
\rho_3 &= \frac{1}{3} U_3 + \frac{1}{6}(U_2 U_1 + U_1 U_2) + \frac{1}{12} {U_1}^3, \\
\rho_4 &= \frac{1}{4} U_4 + \frac{1}{12}(U_3 U_1 + U_1 U_3) + \frac{1}{16} {U_2}^2 + \\
&\quad + \frac{1}{48}({U_1}^2 U_2 + U_2 {U_1}^2) + \frac{1}{36} U_1 U_2 U_1 + \frac{1}{144} {U_1}^4
\end{aligned}$$

Evidently, equation (1.1) defines an invertible, nonlinear mapping from the set of sequences $\{U_n\}$ to the set of sequences $\{\rho_n\}$.

Theorem 1.1. *The inverse mapping is given by*

$$U_n = \sum_{|\mathbf{p}|=n} (-1)^{l-1} \frac{n}{(p_1 + p_2)(p_2 + p_3) \ldots (p_{l-1} + p_l)}\, \rho_{\mathbf{p}} \tag{1.2}$$

where the sum is taken over all compositions $\mathbf{p} = (p_1, \ldots, p_l)$ *of* n*, and* $\rho_{\mathbf{p}} = \rho_{p_1} \rho_{p_2} \cdots \rho_{p_l}$.

What does this theorem have to do with potential functions and spectral residues? Letting $U(z) = U_1 z + U_2 z^2 + \ldots$ and working with the "nearly" self-adjoint equation

$$z^2 \psi''(z) + (1 - \nu) z \psi'(z) + U(z) \psi(z) = 0,$$

we will see below that the corresponding spectral residues are given by equation (1.1). The inversion formula given in Theorem 1.1 is motivated by question 1 above. The proof of the formula will be given below in Section 2.

Before continuing, we should remark that in classical quantum mechanics the potential function $U(z)$ is scalar-valued, and hence the coefficients U_n commute. Likewise the usual application of the method of Frobenius is to series with commuting coefficients. However, formula such as (1.1) and (1.2) involve a sum indexed by compositions, and as such are best treated in a non-commutative setting. Indeed, if the U_n commuted, then the natural form for the inversion formula would be a sum indexed by partitions, i.e. unordered lists of natural numbers, rather than compositions.

The reader should therefore be warned that our discussion of spectral residues will involve both the commutative and the non-commutative settings. In as much as commutativity is an extra assumption, we will try to do as much as possible with non-commuting coefficients. However, much of the discussion, in particular question 2 requires the commutativity assumption. As a consequence, we will find ourselves jumping several times between the two different settings.

As regards question 2, we should first recall that the hypergeometric equation has 3 parameters. Making a suitable change of the parameter variables one of them can be made into a spectral parameter. Taking spectral residues "uses up" this parameter, and as a result the spectral residue sequence consists of polynomials in the remaining two parameters.

We should also observe that the exact-solvability of the hypergeometric potentials is really due to the fact that the hypergeometric equation admits polynomial solutions for certain discrete values of its parameters. This observation has an important consequence: it turns out that the spectral residues of hypergeometric potentials factorize with respect to the two remaining parameters. This in turn leads to interesting combinatorial identities.

Here is an example of this phenomenon associated to the potential function

$$U(z) = \frac{vz}{(1-z)^2} = vz + 2vz^2 + 3vz^3 + \dots ,$$

where the v is a parameter. (N.B. The other hypergeometric parameter was dropped to simplify this example. It will reappear in Section 3, where this sort of factorization is discussed fully.) As per (1.1), the spectral residues are given by

$$\rho_n(v) = \sum_{|\mathbf{p}|=n} \frac{n}{s_{\mathbf{p}}\, s_{\mathbf{p}'}} \left(\prod p_i\right) v^l, \tag{1.3}$$

where the sum is taken over all compositions $\mathbf{p} = (p_1, \dots, p_l)$ of n. Here are the first few of these polynomials:

$$\begin{aligned}
\rho_1 &= v \\
\rho_2 &= v + \frac{1}{2}v^2 \\
\rho_3 &= v + \frac{2}{3}v^2 + \frac{1}{12}v^3 \\
\rho_4 &= v + \frac{3}{4}v^2 + \frac{5}{36}v^3 + \frac{1}{144}v^4
\end{aligned}$$

Routine calculation indicates that the above polynomials can be factored. In fact, we have the following general result.

Theorem 1.2. *The polynomials $\rho_n(v)$ factorize completely over the rationals. Indeed,*

$$n!(n-1)!\,\rho_n = v(v+1\cdot 2)(v+2\cdot 3)\dots(v+(n-1)\cdot n). \tag{1.4}$$

This result can also be appreciated in its converse formulation: the coefficient of v^k of the Pochammer-symbol-like polynomial in the right-hand side of (1.4) is given by the composition sum

$$\sum_{\substack{|\mathbf{p}|=n \\ \ell(\mathbf{p})=k}} \frac{n!}{s_{\mathbf{p}}}\,\frac{n!}{s_{\mathbf{p}'}} \prod_{i=1}^{k} p_i, \tag{1.5}$$

where now the index ranges over only the k-part compositions of n. The proof will be given in Section 3. For now let us stress that the reason underlying this factorization phenomenon is the exact solvability of the corresponding potential $U(z)$. We should also note that this particular $U(z)$ is just a special case of the well-known Eckart potential.

As regards question 3, for now let us limit ourselves to saying that at the level of power series the Darboux transformation is given by a certain invertible, non-linear mapping from a power series

$$U(z) = U_1 z + U_2 z^2 + \dots$$

to a power series

$$\tilde{U}(z) = \tilde{U}_1 z + \tilde{U}_2 z^2 + \dots .$$

The mapping in question is described by a certain composition sum

$$\tilde{U}_n = \sum_{|\mathbf{p}|=n} w(\mathbf{p}) U_{\mathbf{p}},$$

where the weight $w(\mathbf{p})$ is a rational function of the composition elements $p_1, \dots, p_l$ that is too complicated to be written down explicitly.

What is interesting, however, is how easily the Darboux transformation can be described in terms of spectral residues.

Theorem 1.3. *Let U and $\tilde{U}$ be formal powers series with zero constant term. If U and $\tilde{U}$ are related by a Darboux transformation, then the corresponding residue sequence of one is the negative of the other.*

The proof will be given in Section 4. An interesting corollary of this theorem (the proof will again be deferred till Section 4) is the following inversion formula. For each composition $\mathbf{p} = (p_1, \dots, p_{2k+1})$ of odd length set

$$q_{\mathbf{p}} = s_2 s_4 s_6 \dots s_{2k},$$

where again $s_j = p_1 + \dots + p_j$. Next, let $\rho_1, \rho_2, \dots$ be an infinite sequence of non-commuting indeterminates, and for each $n = 1, 2, \dots$ set

$$W_n = \sum_{\substack{|\mathbf{p}|=n \\ \ell(\mathbf{p}) \text{ odd}}} \frac{1}{(p_1+p_2)(p_2+p_3)\dots(p_{l-1}+p_l)}\, \rho_{\mathbf{p}}. \tag{1.6}$$

Corollary 1.4. *The inverse mapping is given by*

$$\rho_n = \sum_{\substack{|\mathbf{p}|=n \\ \ell(\mathbf{p}) \ odd}} (-1)^{\frac{\ell(\mathbf{p})-1}{2}} \frac{1}{q_{\mathbf{p}}\, q_{\mathbf{p}'}} W_{\mathbf{p}}, \tag{1.7}$$

where $W_{\mathbf{p}}$ is an abbreviation for $W_{p_1} W_{p_2} \dots W_{p_l}$.

The organization of the remainder of the paper will mirror the above introductory discussion. In Section 2 we introduce spectral residues, describe how a potential function determines a spectral residue sequence, and give the proof of the inversion formula contained in Theorem 1.1. In Section 3 we will treat the spectral residue sequences of the exactly-solvable, hypergeometric potentials, and derive number of factorization results similar to Theorem 1.2. These results are developed in response to question 2 above, and serve as evidence to the assertion that exact solvability of a given potential function should manifest as factorizability of the corresponding spectral residues. Section 4 we will discuss the Darboux transformation from the point of view of spectral residue sequences. The key contents are the proof of Theorem 1.3, and of the inversion formula of Corollary 1.4. In the final section we briefly survey applications of these identities and inversion formulas to problems in enumeration and to the theory of integrable systems.

2. Spectral residues of Schrödinger operators

We will begin this section by recalling the details of the classic method of Frobenius for power series solutions of a second-order differential equation:

$$z^2\phi''(z) + P(z)z\phi'(z) + Q(z)\phi(z) = 0.$$

We need to generalize the classical method somewhat, and therefore allow $P(z)$ and $Q(z)$ to be power series with coefficients in an arbitrary algebra — not necessarily commutative — over the complex numbers. We want, however, to retain the notion of an indicial equation, and therefore demand that the constant terms of P and Q be complex numbers. It is then easy to show that there exists at least one formal solution of the form

$$z^\nu(1 + \phi_1 z + \phi_2 z^2 + \dots),$$

where $r = \nu$ is a root of the indicial equation

$$r^2 + (P_0 - 1)r + Q_0 = 0.$$

It is well known that there may not be a second such solution if the solutions of the indicial equation differ by an integer.

Since at least one formal solution always exists, no generality will be lost if we assume that one of the roots is zero. We will therefore restrict ourselves to equations

$$\mathcal{T}[\phi] = 0,$$

where $\mathcal{T}$ is a differential operator of the form

$$\mathcal{T} = z^2\partial_{zz} + (1 - \nu)z\partial_z + P(z)z\partial_z + Q(z), \tag{2.8}$$

and both $P(z)$ and $Q(z)$ are power series with zero constant term. This way the indicial equation is $r(r - \nu) = 0$.

If ν is not a positive integer, then there is a unique formal series solution $\phi(z)$ with the initial condition $\phi(0) = 1$. The coefficients are specified by the following recursive relations:

$$n(\nu - n)\phi_n = Q_n + \sum_{j=1}^{n-1}(jP_{n-j} + Q_{n-j})\phi_j.$$

They can also be described in terms of a composition sum:

$$\phi_n = \sum_{|\mathbf{p}|=n} \frac{s_{l-1}P_{p_l} + Q_{p_l}}{n(\nu - n)} \times \dots \times \frac{s_1 P_{p_2} + Q_{p_2}}{s_2(\nu - s_2)} \times \frac{Q_{p_1}}{s_1(\nu - s_1)}, \tag{2.9}$$

where the index ranges over all compositions $\mathbf{p} = (p_1, \dots, p_l)$ of n, and as before $s_j = p_1 + \dots + p_j$.

When ν is a positive integer, then in general there is only one solution; this solution is a series with leading term z^ν. However, under certain circumstances it is possible for a monic series solution to exist as well. To describe when this happens we make the following definition.

Definition 2.1. *Let $\mathcal{T}$ be a second-order differential operator as given in (2.8), but with ν denoting an indeterminate rather than a complex number. Let ϕ_n be the coefficients of the corresponding formal monic series solution. For n a positive integer, we define the n^{th} spectral residue of $\mathcal{T}$ to be the residue of ϕ_n at $\nu = n$.*

Proposition 2.2. *Let $\mathcal{T}$ be as above, and let n be a positive integer. A monic series solution of*

$$\mathcal{T}[\phi] = 0\Big|_{\nu=n}$$

exists if and only if the n^{th} spectral residue of $\mathcal{T}$ is zero.

Proof. The n^{th} spectral residue vanishes if and only if

$$Q_n + \sum_{j=1}^{n-1}(jP_{n-j} + Q_{n-j})\phi_j = 0\Big|_{\nu=n}.$$

If this is the case, then evidently there exists a monic series solution for every possible choice of ϕ_n. □

Another interesting property of spectral residues is their invariance with respect to gauge transformations. In the present context we define a gauge transformation to be the conjugation of a differential operator by an invertible multiplication operator. In particular, given $\mathcal{T}$ of the form shown in (2.8), and a monic power series $\mu = 1 + \mu_1 z + \mu_2 z^2 + \dots$ we obtain another such second-order operator by setting

$$\begin{aligned}\widehat{\mathcal{T}} &= \mu \circ \mathcal{T} \circ \mu^{-1}, \\ &= z^2\partial_{zz} + (1-\nu)z\partial_z + \widehat{P}(z)z\partial_z + \widehat{Q}(z).\end{aligned}$$

Gauge transformations arise naturally in the context of linear differential equations. Indeed, if $\phi(z)$ is a monic solution of $\mathcal{T}[\phi] = 0$, then $\hat{\phi} = \mu\phi$ is evidently a monic solution of $\widehat{\mathcal{T}}[\hat{\phi}] = 0$.

It is also important to note that the presence of the $(1-\nu)z\partial_z$ term in $\mathcal{T}$ will cause the coefficients of $\widehat{Q}(z)$ to depend on the spectral parameter ν. In the subsequent discussion we will therefore broaden our class of operators $\mathcal{T}$ so as to include the possibility that the coefficients of $P(z)$ and $Q(z)$ depend polynomially on ν.

Proposition 2.3. *If two second-order differential operators of the form shown in* (2.8) *are related by a gauge transformation, then all of their spectral residues are equal.*

Proof. Let $\mathcal{T}$, $\widehat{\mathcal{T}}$ and μ be as above, and let $\phi(z)$ and $\hat{\phi}(z)$ be the corresponding monic series solutions. From $\hat{\phi} = \mu\phi$ it follows that

$$\hat{\phi}_n = \phi_n + \text{a linear combination of } \phi_1, \dots \phi_{n-1}.$$

From (2.9) it is evident that the residues of $\phi_1, \dots, \phi_{n-1}$ at $\nu = n$ are all zero. The desired conclusion follows immediately. □

A particularly important type of second-order differential operator is the class of Schrödinger operators (the term Hamiltonian operator will also be used.) In the present setting this will be taken to mean an operator of the form

$$\mathcal{H} = z^2\partial_{zz} + (1-\nu)z\partial_z + U(z), \tag{2.10}$$

where the potential function $U(z) = U_1 z + U_2 z^2 + \dots$ is a formal series with vanishing constant term, and where again we allow the coefficients $U_1, U_2, \dots$ to depend polynomially on ν.

It is more customary to define a Schrödinger operator as a formally self-adjoint operator of the form

$$\widehat{\mathcal{H}} = z^2\partial_z + z\partial_z + U(z) - \lambda, \tag{2.11}$$

where λ is the spectral parameter, and z is related to the physical distance variable x by $z = e^x$. However, the formal eigenfunctions of $\widehat{\mathcal{H}}$ are of the form $z^{\pm\sqrt{\lambda}}$ times a monic series. To get at the monic series directly we conjugate $\widehat{\mathcal{H}}$ by $z^{\nu/2}$, with $\lambda = \nu^2/4$, to obtain an operator in the "nearly" self-adjoint form shown in (2.10).

It is important to note that the class of Schrödinger operators provides a canonical form for second-order differential operators with respect to gauge-transformations.

Proposition 2.4. *Let $\mathcal{T}$ be a second-order operator of the form shown in* (2.8)*. There exists a monic series $\mu(z)$ and a potential function $U(z)$ with zero constant term such that*

$$z^2\partial_{zz} + (1-\nu)z\partial_z + U(z) = \mu \circ \mathcal{T} \circ \mu^{-1}.$$

Proof. In the non-commutative setting it suffices to take

$$\mu_n = \frac{1}{2n}\left(P_n + \sum_{j=1}^{n-1} \mu_{n-j} P_j\right),$$
$$U = \left(\mu Q - z^2\mu'' - (1-\nu)\mu'\right)\mu^{-1},$$

where $P(z)$, $Q(z)$ are the coefficient series of the operator $\mathcal{T}$. In the commutative setting one can use simpler formulas. Indeed, it suffices to introduce an auxiliary function

$$\sigma(z) = \int \frac{P(z)}{2z}, \quad \sigma(0) = 0,$$

and then set

$$\begin{aligned} \mu &= e^{\sigma}, \\ U &= Q - z^2\sigma'' - (1-\nu)z\sigma' - z^2(\sigma')^2. \end{aligned} \tag{2.12}$$

□

Gauge invariance of the spectral residues will be of crucial importance when we consider hypergeometric potentials in the next section. For the remainder of the present section, we will focus our attention on the spectral residues of a Schrödinger operator.

Proposition 2.5. *The n^{th} spectral residue, call it ρ_n, of a Schrödinger operator with potential $U(z) = U_1 z + U_2 z^2 + \dots$ is given by*

$$\rho_n = \sum_{|\mathbf{p}|=n} \frac{n}{s_{\mathbf{p}}\, s_{\mathbf{p}'}}\, U_{\mathbf{p}}.$$

Proof. Taking $P(z) = 0$ and $Q(z) = U(z)$, the desired conclusion follows immediately from (2.9) above. □

From formula (1.1) we see that $\rho_n = n^{-1}U_n +$ lower order terms. It is therefore evident that the sequence of spectral residues $\rho_1, \rho_2, \dots$ completely determines the potential. Surprisingly, the formula for the inverse mapping can be given explicitly. This is formula (1.2) of Theorem 1.1. We conclude the present section by giving a proof of this theorem.

Let $\mathcal{H}$ be a Schrödinger operator of the form shown in (2.10), and let $\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + \dots$ be the formal monic series solution of

$$\mathcal{H}[\phi] = 0.$$

Specializing (2.9) we see that

$$\phi_n = \sum_{|\mathbf{p}|=n} \frac{U_{p_l}}{n(\nu-n)} \times \dots \times \frac{U_{p_2}}{s_2(\nu-s_2)} \times \frac{U_{p_1}}{s_1(\nu-s_1)}. \tag{2.13}$$

For all pairs of positive integers k and n with $k \le n$, we let $\rho_{n,k}$ denote the residue of ϕ_n at $\nu = k$. Thus, $\rho_{n,n}$ is just another way to refer to the n^{th} spectral residue ρ_n.

We are now able to write

$$\phi_n = \frac{\rho_{n,n}}{\nu - n} + \frac{\rho_{n,n-1}}{\nu - n + 1} + \ldots + \frac{\rho_{n,1}}{\nu - 1}. \tag{2.14}$$

Next, we evaluate $\rho_{n,k}$ in terms of the coefficients of $U(z)$. From (2.13) we obtain

$$\rho_{n,k} = \sum_{\substack{|\mathbf{p}|=k \\ |\mathbf{q}|=n-k}} \frac{U_{q_m} \ldots U_{q_1}}{(k+t_m)(-t_m)\ldots(k+t_1)(-t_1)} \times \frac{U_{p_l} \ldots U_{p_1}}{k \ldots (k-s_2)s_2(k-s_1)s_1},$$

where the sum is indexed by all compositions $\mathbf{p} = (p_1, \ldots, p_l)$ of k and all compositions $\mathbf{q} = (q_1, \ldots, q_m)$ of $n-k$, and where $s_j = p_1 + \ldots + p_j$ and $t_j = q_1 + \ldots + q_j$. It is evident by inspection that the above sum can be factored, and indeed that

$$\rho_{n,k} = \phi_{n-k}\Big|_{\nu=-k} \times \rho_k.$$

Using (2.14) once again we obtain

$$\rho_{n,k} = -\left(\frac{\rho_{n-k}}{k+n} + \frac{\rho_{n-k,n-k-1}}{k+n-1} + \ldots + \frac{\rho_{n-k,1}}{k+1}\right)\rho_k. \tag{2.15}$$

Now ϕ is a solution of $\mathcal{H}[\phi] = 0$ if and only if

$$n(\nu - n)\phi_n = U_n + \sum_{j=1}^{n-1} U_{n-j}\phi_j.$$

Substituting (2.14) into the left hand side, and evaluating both sides at $\nu = \infty$ we obtain

$$U_n = n\left(\rho_n + \rho_{n,n-1} + \ldots + \rho_{n,1}\right).$$

This equation together with (2.15) gives a recursive specification of U_n in terms of $\rho_1, \ldots, \rho_n$. Rewriting this as a composition sum we obtain the desired formula (1.2). □

3. Spectral residues of exactly solvable potentials

We have already mentioned in the introduction that the spectral residue sequence is sensitive to the exact solvability of the underlying potential. In the present section we will discuss this phenomenon in some detail, and use it to derive a number of factorization identities similar to the identity described in Theorem 1.2 above.

The key to the whole matter is the invariance of spectral residues with respect to gauge transformations — see Proposition 2.3 above. Hypergeometric functions also play an important role. Indeed the exact solvability of the potentials that we will be considering here is based on the fact that the corresponding Schrödigner operators are gauge

equivalent to the hypergeometric operator. Certain values of the hypergeometric parameters cause the hypergeometric series to truncate, which means that for those values of the parameters almost all the spectral residues vanish. Thus, when we consider spectral residues as polynomials in the hypergeometric parameters we obtain factorizations of said polynomials.

Let us also remark that in the present section we can safely restrict ourselves to power series with complex coefficients. We will consider non-commutative coefficients again in section 4.

Three families of exactly-solvable potentials will be treated here: the Eckart [12], the Pöschl-Teller [13], and the Morse [14] potentials. The first two of these have eigenfunctions related to Gaussian hypergeometric functions; in the latter case one needs to use confluent hypergeometric functions.

We will discuss each of these families in turn. In each case we will first show how the Schrödinger's equation is related to the hypergeometric or the confluent hypergeometric equation. Next we will consider the corresponding spectral residues and see that in each case there is complete factorization in terms of the potential parameters. Finally, we will list some interesting factorization identities that arise for each of the families for certain specialized values of the parameters.

3.1. **Eckart Potentials.** The Eckart potentials are functions of the form

$$U = \frac{u\,\mathrm{e}^x}{1-\mathrm{e}^x} + \frac{v\,\mathrm{e}^x}{(1-\mathrm{e}^x)^2},$$

where x is the physical distance variable, and u and v are parameters. Rewriting the Hamiltonian, $\mathcal{H}$, in terms of $z = \mathrm{e}^x$ we get

$$\mathcal{H} = z^2\partial_{zz} + (1-\nu)z\partial_z + \frac{uz}{1-z} + \frac{vz}{(1-z)^2}, \tag{3.16}$$

where the spectral parameter ν has been encoded into the first-order part of the operator — see the preceding Section for a discussion of this matter.

Let us now look for the connection between Eckart potentials and hypergeometric functions. The hypergeometric series,

$$F(\alpha,\beta,\gamma;z) = F_0 + F_1 z + F_2 z^2 + \dots\,,$$

can be defined as the solution of the hypergeometric differential equation:

$$\mathcal{G}(F) = 0, \quad F(0) = 1, \tag{3.17}$$

where $\mathcal{G}$ is the familiar second order differential operator

$$\mathcal{G} = z(1-z)\partial_{zz} + (\gamma - (\alpha+\beta+1)z)\,\partial_z - \alpha\beta. \tag{3.18}$$

In order to work with spectral residues we need to somehow convert the hypergeometric operator into an operator of the form $z^2\partial_{zz}+$ lower order terms. An obvious way to do this is to left-multiply by $z/(1-z)$. To wit, set

$$\begin{aligned}\mathcal{G}_1 &= \frac{z}{1-z}\,\mathcal{G} \\ &= z^2\partial_{zz} + (1-\nu)z\partial_z - \frac{z}{1-z}\,[(\alpha+\beta+\nu)z\partial_z + \alpha\beta]\end{aligned}$$

where $\nu = 1-\gamma$ is the required spectral parameter. The upshot of all this is that one can equally well specify the hypergeometric series F as the solution of

$$\mathcal{G}_1(F) = 0, \quad F(0) = 1.$$

In Proposition 2.4 of the preceding section we noted that every second-order differential operator is gauge-equivalent to a Schrödinger operator. Following the method described there, we conjugate $\mathcal{G}_1$ by a multiplication operator $\mu = e^\sigma$ where

$$\begin{aligned}\sigma(z) &= -\frac{1}{2}(\alpha+\beta+\nu)\int \frac{1}{1-z} \\ &= \frac{1}{2}(\alpha+\beta+\nu)\,\log(1-z); \\ \mu(z) &= (1-z)^{(\alpha+\beta+\nu)/2}.\end{aligned}$$

The end-result of the gauge-transformation is the Eckart Hamiltonian shown in (3.16). Using (2.12), an elementary calculation will show that the potential parameters are related to the hypergeometric parameters by

$$\begin{aligned}u &= \frac{1}{4}\,(\alpha-\beta-\nu)(\alpha-\beta+\nu), \\ v &= \frac{1}{4}\,(\alpha+\beta+\nu)(2-\alpha-\beta-\nu).\end{aligned} \tag{3.19}$$

Turning next to spectral residues, we expand the Eckart potential in a series with respect to z:

$$U = \sum_k (u+kv)z^k.$$

In Proposition 2.5 we proved that the spectral residues of a Schrödinger equation are given by formula (1.1). From this formula and from the

above expansion we obtain the expression for the spectral residues of the Eckart potentials:

$$\rho_n(u,v) = n \sum_{|\mathbf{p}|=n} \frac{\prod_i (u+vp_i)}{s_{\mathbf{p}}\, s_{\mathbf{p}'}}, \tag{3.20}$$

where as usual the index ranges over all compositions $\mathbf{p} = (p_1, \dots, p_l)$ of n. We have arrived at the following result.

Proposition 3.1. *In the even case, $n = 2m$, one has*

$$\rho_n(u,v) = K \prod_{i=0}^{m-1} \left[(u+v+a_{n,i})(u+v+a_{n,i+1}) + b_{n,i}^2 v \right], \tag{3.21}$$

where $K = \frac{1}{n!(n-1)!}$, $a_{n,i} = i(n-i)$ and $b_{n,i} = n-1-2i$. In the odd case, $n = 2m+1$, one has

$$\rho_n(u,v) = K(u+v+a_{n,m}) \prod_{i=0}^{m-1} \left[(u+v+a_{n,i})(u+v+a_{n,i+1}) + b_{n,i}^2 v \right]. \tag{3.22}$$

Proof. We saw above that the eigenfunctions of the Eckart Hamiltonian are the product of a power of $(1-z)$ and of a hypergeometric series. Hence by the gauge-invariance of spectral residues as asserted in Proposition 2.3 above, the spectral residues of the Eckart Hamiltonian can just as well be calculated by taking residues of the coefficients of the hypergeometric series.

These are of course the well-known:

$$F_n = \frac{[\alpha]^n [\beta]^n}{n!\,[\gamma]^n}, \tag{3.23}$$

where $[x]^n$ denotes the rising factorial $x(x+1)\dots(x+n-1)$. Calculating the residue of F_n at $\nu = n$ (recall that $\gamma = 1-\nu$), and using the gauge-invariance of ρ_n we obtain

$$\rho_n(u,v)\Big|_{\nu=n} = (-1)^n \frac{[\alpha]^n [\beta]^n}{n!(n-1)!}. \tag{3.24}$$

From (3.19) it follows that

$$(\alpha+i)(\beta+i)(\alpha+n-1-i)(\beta+n-1-i) = \\ (u+v+a_{n,i})(u+v+a_{n,i+1}) + b_{n,i}^2 v \,\Big|_{\nu=n}, \tag{3.25}$$

Another calculation will show that if n is odd, say $n = 2m+1$, then

$$-(\alpha+m)(\beta+m) = u+v+a_{n,m}\,\Big|_{\nu=n}.$$

This observation explains why the $(-1)^n$ factor in (3.24) can be discarded. The proposition now follows immediately. □

To conclude the discussion of the Eckart potentials let us mention some identities that come about as a consequence of Proposition 3.1. The first of these was mentioned in Theorem 1.2 of the introduction. For the proof, we simply set $u = 0$ in (3.21) and (3.22) above, and note that

$$(v + a_{n,i})(v + a_{n,i+1}) + b_{n,i}^2 v = (v + i(i+1))\,(v + (n-i-1)(n-i))\,.$$

A similar factorization identity can also be obtained by setting $u = -v$ in (3.22) and (3.21). In order to describe this identity, for positive integers $k \leq n$ let us set

$$c_{n,k} = \sum_{\substack{|\mathbf{p}|=n \\ \ell(\mathbf{p})=k}} \frac{n!}{s_{\mathbf{p}}} \frac{n!}{s_{\mathbf{p}'}} \prod_i (p_i - 1) \tag{3.26}$$

Corollary 3.2. *In the even case, $n = 2m$, the above numbers $c_{n,k}$ are given by the coefficient of x^k in*

$$\prod_{i=0}^{m-1} \Big((n-1-2i)^2 x + i(i+1)(n-1-i)(n-i) \Big).$$

An analogous statement holds in the odd case, $n = 2m+1$, but with an additional factor of $m(m+1)$ appended to the above expression.

3.2. **Pöschll-Teller Potentials.** Turning next to the Pöschl-Teller potentials, these are functions of the form

$$\frac{u}{\cosh^2(x/2)} + \frac{v}{\sinh^2(x/2)},$$

where again x is the physical distance variable, and u, v are parameters. Rewriting the Hamiltonian, $\mathcal{H}$, in terms of $z = \mathrm{e}^x$ we get

$$\mathcal{H} = z^2 \partial_{zz} + (1-\nu) z \partial_z + \frac{uz}{(1+z)^2} + \frac{vz}{(1-z)^2}. \tag{3.27}$$

Let us next consider how $\mathcal{H}$ can be related to the hypergeometric operator $\mathcal{G}$. To begin, let us rewrite the latter using ζ as the independent variable:

$$\mathcal{G} = \zeta(1-\zeta)\partial_{\zeta\zeta} + (\gamma - (\alpha+\beta+1)\zeta)\,\partial_\zeta - \alpha\beta.$$

We note that a change of variables $\cosh x = 2\zeta - 1$ will transform the leading term of $\mathcal{G}$ into $-\partial_{xx}$. Since we would like the leading term to be $-z^2\partial_{zz}$ (the minus sign doesn't really matter), we employ $z = e^x$

as the independent variable. To make the change of variables we note that

$$\zeta = \frac{(z+1)^2}{4z}, \quad \partial_\zeta = \frac{4z^2}{z^2-1}\partial_z. \tag{3.28}$$

In this way we obtain

$$-\mathcal{G} = z^2\partial_{zz} + (1-\alpha-\beta)z\partial_z + \left((2\gamma-1)\frac{z}{z+1} + (2\alpha+2\beta+1-2\gamma)\frac{z}{z-1}\right) z\partial_z + \alpha\beta$$

The roots of the indicial equation are α and β, and therefore, in order to obtain solutions that are monic power series we make a gauge-transformation:

$$\begin{aligned}\mathcal{G}_1 &= -z^{-\alpha} \circ \mathcal{G} \circ z^{\alpha} \\ &= z^2\partial_{zz} + (1+\alpha-\beta)z\partial_z + \\ &\quad + \left((2\gamma-1)\frac{z}{z+1} + (2\alpha+2\beta+1-2\gamma)\frac{z}{z-1}\right)(z\partial_z + \alpha)\end{aligned}$$

Employing the abbreviations

$$\begin{aligned}\lambda &= \gamma - 1/2, \\ \mu &= \alpha + \beta + 1/2 - \gamma, \\ \nu &= \beta - \alpha, \\ w(z) &= \frac{\lambda z}{z+1} + \frac{\mu z}{z-1}\end{aligned} \tag{3.29}$$

the above operator can be more compactly rewritten as

$$\mathcal{G}_1 = z^2\partial_{zz} + (1-\nu)z\partial_z + 2w(z)z\partial_z + (\lambda+\mu-\nu)w(z)$$

From there, the following gauge transformation will give the Pöschl-Teller Hamiltonian:

$$\mathcal{H} = \left((1+z)^{\lambda}(1-z)^{\mu}\right) \circ \mathcal{G}_1 \circ \left((1+z)^{-\lambda}(1-z)^{-\mu}\right),$$

provided the potential parameters are related to the hypergeometric parameters by

$$u = \lambda^2 - \lambda, \quad v = \mu - \mu^2.$$

Turning next to the eigenfunctions of $\mathcal{H}$ we encounter a difficulty: we do not get a power series if we substitute $\zeta = (z+1)^2/(4z)$ into the usual hypergeometric series $F(\alpha,\beta,\gamma;\zeta)$. Note though that $4z/(1+z)^2$ is analytic in z, and so we can overcome this difficulty by employing a solution of the hypergeometric equation that is analytic, or nearly so, in ζ^{-1}. Just such a solution can be found on Kummer's list of 24

solutions to the hypergeometric equation — see Section 2.9 of [5]. The solution to $\mathcal{G}(\varphi) = 0$ we require is

$$\varphi(\zeta) = \zeta^{-\alpha} F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta; \zeta^{-1}).$$

Let us summarize this observation in the following manner.

Lemma 3.3. *The unique series solution $\phi = \phi(z)$ to*

$$\mathcal{G}_1(\phi) = 0, \quad \phi(0) = 1,$$

is given by

$$\phi(z) = (1+z)^{\nu-\lambda-\mu} F\left(\frac{\mu+\lambda-\nu}{2}, \frac{\mu-\lambda+1-\nu}{2}, 1-\nu; \frac{4z}{(1+z)^2}\right).$$

We are now in a position to factorize the spectral residues of the Pöschl-Teller potential. Expanding the Pöschl-Teller potential in a power series we obtain

$$U(z) = \sum_{k \text{ even}} k u_0 \, z^k + \sum_{k \text{ odd}} k u_1 \, z^k,$$

where $u_0 = v - u$ and $u_1 = v + u$. Hence, by Proposition 2.5 the n^{th} spectral residue, is given by

$$\rho_n = n \sum_{|\mathbf{p}|=n} \frac{\prod_i p_i}{s_{\mathbf{p}} s_{\mathbf{p}'}} \, u_1^{o(\mathbf{p})} u_0^{e(\mathbf{p})}, \tag{3.30}$$

where $o(\mathbf{p})$ is the number of odd elements of a composition $\mathbf{p} = (p_1, \ldots, p_l)$, and $e(\mathbf{p})$ is the number of even elements.

Proposition 3.4. *In the even case, $n = 2m$, one has*

$$\rho_n = K \prod_{\substack{0<k<n \\ k \text{ odd}}} \Big(u_1^2 + 2k^2 u_0 + k^4 - k^2 \Big), \tag{3.31}$$

where $K = \frac{1}{n!(n-1)!}$. In the odd case, $n = 2m+1$, one has

$$\rho_n = K u_1 \prod_{\substack{0<k<n \\ k \text{ even}}} \Big(u_1^2 + 2k^2 u_0 + k^4 - k^2 \Big), \tag{3.32}$$

Proof. Let $\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + \ldots$ be the unique series solution to

$$\mathcal{G}_1(\phi) = 0, \quad \phi(0) = 1.$$

By Lemma 3.3, if $(\mu + \lambda - \nu)/2 = -j$, $j \in \mathbb{N}$, then $\phi(z)$ is $(1 + z)^{\nu-\lambda-\mu}$ times a j^{th} degree polynomial of $z/(z+1)^2$ whose coefficients are polynomials of λ, μ, ν over $(1-\nu)(2-\nu)\ldots(j-\nu)$. Hence for all $n > j$ the residue of ϕ_n at $\nu = n$ must have $\mu + \lambda + 2j - n$ as a factor. By the gauge invariance of spectral residues, the same must be true for

ρ_n. The same reasoning also applies to show that $\lambda - \mu - 1 - 2j + n$ is a factor of ρ_n for all $j = 0, 1, \ldots, n-1$.

We have now shown that ρ_n, as a function of λ and μ, is some factor K times

$$\prod_{j=0}^{n-1}(\mu + \lambda + 2j - n)(\lambda - \mu - 1 - 2j + n).$$

Now

$$u_0 = v - u = \mu + \lambda - \mu^2 - \lambda^2,$$
$$u_1 = v + u = \mu - \lambda - \mu^2 + \lambda^2 = (\lambda - \mu)(\mu + \lambda + 1),$$

and hence for all k

$$u_1^2 + 2k^2u_0 + k^4 - k^2 = \\ (\lambda + \mu + k - 1)(\lambda + \mu - k - 1)(\lambda - \mu + k)(\lambda - \mu - k)$$

Using (3.30), a degree count, and an examination of the coefficient of u_1^n we see that $K = \frac{1}{n!(n-1)!}$. The proposition now follows immediately. □

Here are some interesting identities arising from the preceding factorization.

Corollary 3.5. *The coefficient of x^k in*

$$(x + 2^4)(x + 4^4)(x + 6^4)\ldots(x + (2m)^4)$$

is given by the following composition sum:

$$\sum_{\substack{|\mathbf{p}|=2m+1 \\ o(\mathbf{p})=2k+1}} \frac{n!}{s_{\mathbf{p}}} \frac{n!}{s_{\mathbf{p}'}} \frac{\prod_i p_i}{2^{e(\mathbf{p})}} \tag{3.33}$$

Proof. Upon setting $x = u_1^2$, $u_0 = 1/2$, the above identity follows from the odd, $n = 2m + 1$, case of Proposition 3.4. □

If we use the same summand as above, but restrict ourselves to compositions without even numbers we obtain the coefficients of a closely related product.

Corollary 3.6. *The coefficient of x^k in*

$$(x + 2^4 - 2^2)(x + 4^4 - 4^2)(x + 6^4 - 6^2)\ldots(x + (2m)^4 - (2m)^2)$$

is given by the following compositions sum:

$$\sum_{\substack{|\mathbf{p}|=2m+1 \\ o(\mathbf{p})=2k+1 \\ e(\mathbf{p})=0}} \frac{n!}{s_{\mathbf{p}}} \frac{n!}{s_{\mathbf{p}'}} \prod_i p_i$$

Proof. Upon setting $x = u_1^2$, $u_0 = 0$, the above identity follows from the odd, $n = 2m + 1$, case of Proposition 3.4. □

Identities featuring k^4 and $k^4 - k^2$, with k odd, can be obtained in an analogous manner.

3.3. **Morse Potentials.** Finally, let us consider Morse potentials; these are functions of the form

$$ue^x + ve^{2x},$$

where x is the physical distance variable, and u, v are parameters. Rewriting the Hamiltonian, $\mathcal{H}$, in terms of $z = \mathrm{e}^x$ we get

$$\mathcal{H} = z^2\partial_{zz} + (1-\nu)z\partial_z + uz + vz^2, \tag{3.34}$$

The eigenfunctions of the above operator can be related to confluent hypergeometric functions by a gauge transformation. The confluent hypergeometric series can be defined as the solution of the confluent hypergeometric differential equation:

$$\mathcal{J}(\phi) = 0, \quad \phi(0) = 1,$$

where $\mathcal{J}$ is the second order differential operator

$$\mathcal{J} = z\partial_{zz} + (\gamma - z)\partial_z + \alpha$$

We need to introduce an extra scaling parameter, ω, stemming from the change of variables $z \mapsto -2\omega z$. Now, the relevant operator is given by

$$\mathcal{J} = z\partial_{zz} + (\gamma + 2\omega z)\partial_z + 2\omega\alpha,$$

and the solution,

$$\phi(\alpha, \gamma, \omega; z) = \phi_0 + \phi_1 z + \phi_2 z^2 + \dots,$$

of $\mathcal{J}[\phi] = 0$ depends on 3 parameters.

The equation $\mathcal{J}[\phi] = 0$ is equivalent to a certain relation between ϕ_n and ϕ_{n+1}, and yields the following modification of the familiar formula for the coefficients of ϕ:

$$\phi_n = \frac{(-2)^n \omega^n [\alpha]^n}{n!\,[\gamma]^n}.$$

Proceeding as we did in the case of the Eckart potentials we modify $\mathcal{J}$ so as to obtain an operator with leading term $z^2\partial_{zz}$:

$$\mathcal{J}_1 = z\mathcal{J} = z^2\partial_{zz} + (1-\nu)z\partial_z + 2\omega z^2\partial_z + 2\omega\alpha z,$$

where again we employ $\nu = 1 - \gamma$. Next we convert $\mathcal{J}_1$ to (nearly) self-adjoint form by means of a gauge transformation with gauge factor $\mu = e^{\sigma}$, where

$$\sigma = \int^z \omega = \omega z.$$

Hence, $\mu = e^{\omega z}$. The result is the Hamiltonian operator shown in (3.34), where the potential parameters are related to the hypergeometric parameters by

$$\begin{aligned} u &= \omega(\nu + 2\alpha - 1), \\ v &= -\omega^2. \end{aligned} \tag{3.35}$$

To obtain the corresponding spectral residues we again employ (1.1):

$$\rho_n(u, v) = n \sum_{\substack{|\mathbf{p}|=n \\ p_i \in \{1,2\}}} \frac{u^{o(\mathbf{p})}\, v^{e(\mathbf{p})}}{s_{\mathbf{p}}\, s_{\mathbf{p}'}},$$

where the index now ranges over compositions $\mathbf{p} = (p_1, \ldots, p_l)$ of n consisting of 1 and 2 only, and where it is necessary to recall that $o(\mathbf{p})$ denotes the number of odd elements of a composition $\mathbf{p} = (p_1, \ldots, p_l)$, and $e(\mathbf{p})$ the number of even elements.

Proposition 3.7. *In the even case, $n = 2m$, one has*

$$\rho_n(u, v) = K \prod_{\substack{0<k<n \\ k \text{ odd}}} (u^2 + k^2 v), \tag{3.36}$$

where $K = \frac{1}{n!(n-1)!}$. In the odd case, $n = 2m + 1$, one has

$$\rho_n(u, v) = K u \prod_{\substack{0<k<n \\ k \text{ even}}} (u^2 + k^2 v), \tag{3.37}$$

Proof. We proceed in the same way as in the proof of Proposition (3.1). By the gauge-invariance of spectral residues, ρ_n is equal to the n^{th} spectral residue of the confluent hypergeometric operator. Taking the residue of ϕ_n at $\nu = n$ yields

$$\frac{2^n \omega^n [\alpha]^n}{(n!)^2}.$$

Next, using (3.35) we see that in the first case, where $n = 2m, k = 2i+1$, we have

$$u^2 + k^2 v \,\Big|_{\nu=n} = 4\,\omega^2(\alpha + m + i)(\alpha + m - i - 1).$$

In the second case, where $n = 2m + 1$, $k = 2i$ we have

$$u^2 + k^2 v \Big|_{\nu=n} = 4\,\omega^2(\alpha + m + i)(\alpha + m - i),$$
$$u \Big|_{\nu=n} = 2\,\omega(\alpha + m).$$

The proposition now follows immediately. □

Setting $v = 1$ in the above Proposition immediately yields the following identities:

Corollary 3.8. *The coefficient of x^k in*

$$(x + 2^2)(x + 4^2)(x + 6^2)\dots(x + (2m)^2)$$

is given by the following composition sum:

$$\sum_{\substack{|\mathbf{p}|=2m+1 \\ o(\mathbf{p})=2k+1 \\ p_i \in \{1,2\}}} \frac{n!}{s_{\mathbf{p}}} \frac{n!}{s_{\mathbf{p}'}}$$

The coefficient of x^k in

$$(x + 1^2)(x + 3^2)(x + 5^2)\dots(x + (2m-1)^2)$$

is given by the following composition sum:

$$\sum_{\substack{|\mathbf{p}|=2m \\ o(\mathbf{p})=2k \\ p_i \in \{1,2\}}} \frac{n!}{s_{\mathbf{p}}} \frac{n!}{s_{\mathbf{p}'}}$$

4. The Darboux transformation

We begin this section by describing the Darboux transformation for linear, second-order differential equations. Let us consider a self-adjoint, second-order operator

$$\mathcal{H} = \partial_{xx} + U(x),$$

and ask whether it can be factored into a product of first order operators. It is not hard to see that the answer is "yes", and that the factorization must be of the form

$$\mathcal{H} = (\partial_x - W)(\partial_x + W),$$

where the function $W(x)$ — we will refer to it as a prepotential — is related to the potential by $U = W_x - W^2$. Reversing the order of the above factorization we obtain another self-adjoint operator:

$$\widetilde{\mathcal{H}} = (\partial_x + W)(\partial_x - W) = \partial_{xx} + \tilde{U},$$

where now $\tilde{U} = -W_x - W^2$. Knowing U one has to solve a Ricatti equation in order to obtain W. Alternatively, one can describe the prepotential as $W = -\partial_x(\log\psi)$, where $\psi(x)$ is a solution of $\mathcal{H}(\psi) = 0$.

The process of going from $U(x)$ to $\tilde{U}(x)$ is called the Darboux transformation. It is evident, that at the level of the prepotential, the Darboux transformation corresponds to a negation, and that therefore the Darboux transform of $\tilde{U}$ takes us back to U.

There is a close relationship between the (formal) eigenfunctions of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. Indeed, one can write

$$\begin{aligned}(\partial_x + W)(\partial_x - W)(\partial_x + W) &= (\partial_x + W)\mathcal{H} \\ &= \widetilde{\mathcal{H}}(\partial_x + W),\end{aligned} \tag{4.38}$$

From here, suppose that ψ is a formal eigenfunction of $\mathcal{H}$, i.e. that $\mathcal{H}(\psi) = \lambda\psi$. Setting $\tilde{\psi} = \psi_x + W\psi$ we employ (4.38) to conclude that $\widetilde{\mathcal{H}}(\tilde{\psi}) = \lambda\tilde{\psi}$.

In order to apply the Darboux transformation in a truly analytic setting, one has to take into account factors like square-summability of the eigenfunctions, and smoothness of the potential. Here, we are more interested in formal properties of the Darboux transformation. Moreover, our approach is based on power series in $z = e^x$, rather than on functions of x.

Let us therefore adapt the notion of a Darboux transformation to the present setting. We consider a second order operator

$$\mathcal{H} = z^2\partial_{zz} + (1-\nu)z\partial_z + U,$$

where $U(z)$ is a formal power series with vanishing constant term. We do not assume that the coefficients U_n commute. Owing to the fact that $U(z)$ has zero constant term, $\mathcal{H}$ cannot be factored into a product of first order operators. In order to obtain a factorization we must add a constant. To wit,

$$\mathcal{H} + \nu^2/4 = (z\partial_z - \nu/2 - W)(z\partial_z - \nu/2 + W),$$

where $W(z) = W_1 z + W_2 z^2 + \dots$ is another formal power series related to the potential by

$$U = zW_z - W^2.$$

At the level of coefficients this is equivalent to an infinite number of graded relations:

$$U_n = nW_n - \sum_{i=1}^{n-1} W_i W_{n-i}.$$

Note that $W_n = n^{-1}\,U_n+$ lower order terms, and hence the above relations can be recursively inverted, so that if we know the coefficients of U, we can calculate the coefficients of W.

Reversing the above factorization we arrive at the partner operator

$$\widetilde{\mathcal{H}} = z^2\partial_{zz} + (1-\nu)z\partial_z + \tilde{U},$$

were $\tilde{U} = -zW_z - W^2$. Here are the first few relations that describe the formal Darboux transformation:

$$\tilde{U}_1 = -U_1, \quad \tilde{U}_2 = -2\,U_1^2 - U_2, \quad \tilde{U}_3 = -2\,U_1^3 - 2\,U_1U_2 - U_3$$

It does not seem possible to obtain a closed sum formula for the above relations.

Next, let $\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + \dots$ be the formal monic solution of

$$\mathcal{H}(\phi) = 0.$$

We saw in Section 2 that the coefficients of ϕ will be certain rational functions in ν. We set

$$\tilde{\phi} = z\phi_z + (-\nu/2 + W)\phi,$$

and note that since

$$(\widetilde{\mathcal{H}} + \nu^2/4)(z\partial_z - \nu/2 + W) = (z\partial_z - \nu/2 + W)(\mathcal{H} + \nu^2/4),$$

we can conclude that

$$\widetilde{\mathcal{H}}(\tilde{\phi}) = 0.$$

We are now ready to prove Theorem 1.3. Note that

$$\psi = -2\,\nu^{-1}\tilde{\phi}$$

is the unique monic solution of

$$\widetilde{\mathcal{H}}(\psi) = 0.$$

Hence the n^{th} spectral residue of $\tilde{U}$, call it $\tilde{\rho}_n$ is given by

$$\tilde{\rho}_n = -\frac{2}{n}\,\mathrm{Res}(\tilde{\phi}_n; \nu = n).$$

Since

$$\tilde{\phi}_n = (n - \nu/2)\phi_n + \text{ lower order terms},$$

it follows that

$$\tilde{\rho}_n = -\frac{2}{n}\left(n - \frac{n}{2}\right)\rho_n = -\rho_n,$$

where ρ_n is the n^{th} spectral residue of U. The theorem is proved.

Finally, let us see how the relation between the spectral residues and the prepotential leads us to the inversion formulas (1.6) and (1.7). With Theorem 1.1 at our disposal it is not difficult to describe W in terms of the ρ_n.

Proposition 4.1.

$$W_n = \sum_{\substack{|\mathbf{p}|=n \\ \ell(\mathbf{p}) \text{ odd}}} \frac{1}{(p_1+p_2)(p_2+p_3)\dots(p_{l-1}+p_l)}\,\rho_{\mathbf{p}}.$$

Proof. Note that $2zW_z = U - \tilde{U}$, or equivalently that $W_n = (U_n - \tilde{U}_n/(2n)$. Furthermore, Theorems 1.1 and 1.3 imply that

$$U_n = \sum_{|\mathbf{p}|=n} \frac{(-1)^{l-1}\,n}{(p_1+p_2)(p_2+p_3)\dots(p_{l-1}+p_l)}\,\rho_{\mathbf{p}}$$

$$\tilde{U}_n = \sum_{|\mathbf{p}|=n} \frac{-n}{(p_1+p_2)(p_2+p_3)\dots(p_{l-1}+p_l)}\,\rho_{\mathbf{p}}$$

The desired conclusion now follows immediately. □

The inverse relation, i.e. a formula for the spectral residues in terms of the W_n is shown in equation (1.7). Let us prove it now. Let ϕ and $\tilde{\phi}$ denote, respectively the monic solutions of $\mathcal{H}(\phi) = 0$ and $\widetilde{\mathcal{H}}(\tilde{\phi}) = 0$. Alternatively, we could consider ϕ and $\tilde{\phi}$ together as the solutions of the following coupled first-order equations:

$$\begin{aligned} z\phi_z - \frac{\nu}{2}\,\phi + W\phi &= -\frac{\nu}{2}\,\tilde{\phi} \\ z\tilde{\phi}_z - \frac{\nu}{2}\,\tilde{\phi} - W\tilde{\phi} &= -\frac{\nu}{2}\,\phi \\ \phi(0) = \tilde{\phi}(0) &= 1 \end{aligned}$$

Introducing sum and difference variables:

$$\sigma = (\phi + \tilde{\phi})/2, \qquad \delta = (\phi - \tilde{\phi})/2,$$

we obtain

$$\begin{aligned} z\sigma_z &= -W\delta, & \sigma(0) &= 1, \\ z\delta_z - \nu\delta &= -W\sigma, & \delta(0) &= 0 \end{aligned}$$

In terms of power series coefficients we have

$$\begin{aligned} n\sigma_n &= -W_{n-1}\delta_1 - \dots - W_1\delta_{n-1} \\ (n-\nu)\delta_n &= -W_n - W_{n-1}\sigma_1 - \dots - W_1\sigma_{n-1} \end{aligned}$$

Now

$$\operatorname{Res}(\phi_n; \nu = n) = \rho_n,$$

and by Theorem 1.3,

$$\operatorname{Res}(\tilde{\phi}_n; \nu = n) = -\rho_n.$$

Consequently,

$$\rho_n = \mathrm{Res}(\delta_n; \nu = n).$$

It is evident that given $W_1, W_2, \ldots$ the above relations can be solved for σ_n and It is also evident that each σ_n will be a sum of products of even numbers of W_n and that each δ_n will be a sum of products of odd numbers of W_n. Indeed, solving the above relations for δ_n we obtain

$$\delta_n = \sum_{\substack{|\mathbf{p}|=n \\ \ell(\mathbf{p}) \text{ odd}}} \frac{W_l}{\nu - n} \times \ldots \times \frac{W_{p_4}}{(-s_4)} \times \frac{W_{p_3}}{\nu - s_3} \times \frac{W_{p_2}}{(-s_2)} \times \frac{W_{p_1}}{\nu - s_1},$$

where as usual $\mathbf{p} = (p_1, \ldots, p_l)$ is a composition of n, and $s_j = p_1 + \ldots + p_j$. Taking the residue of the right hand side at $\nu = n$, equation (1.7) follows immediately.

5. Applications and Conclusion

The above discussion indicates how the notion of a spectral residue can be used to give a formal series treatment of differential equations with spectral parameter. Our focus has been on second-order, one-dimensional problems. This is a well-understood theory with a large selection of techniques for exact solutions. It has been our aim to demonstrate that such techniques lead to useful information once we make the jump to the formal domain. The author feels that the application of the spectral residue methodology to more general classes of eigenvalue problems, e.g. non-linear and higher-dimensional linear problems, will yield additional combinatorial results.

In conclusion, let us also indicate some applications for the identities and the inversion formulas developed here. We will consider two topics: enumeration and integrable systems. An application to shape-invariant potentials has been reported on in [15]. The author hopes that the existence of such applications will serve as an incentive for a further investigation of spectral residues.

5.1. **Enumeration problems.** Consider a selection of k triples of numbers (a, b, c) with $1 \leq a < b < c \leq n$. Alternatively we could talk about a $3 \times k$ rectangular tableaux whose entries are numbers from 1 to n (repetitions permitted) with the constraint that the columns be arranged in ascending order. A natural question in this setting is the following. If the k triples are selected in a uniformly random manner, what is more likely: a multiple occurrence of a number in the second row, or a multiple occurrence in the third row. The answer is that duplication is more probable in the third row (or indeed in the first row by consideration of symmetry) than in the second row.

This fact is straightforward consequence of the identity shown in (1.4). To see why let us remark that $\binom{m-1}{2}$ is the number of triples with m as the third entry. Therefore the generating function that counts tableaux with distinct entries in the third row is

$$F(x) = \left(1 + \frac{1 \cdot 2}{2}\, x\right) \left(1 + \frac{2 \cdot 3}{2}\, x\right) \dots \left(1 + \frac{n(n-1)}{2}\, x\right).$$

This is more or less the right hand side of equation (1.4). Next we note that $m(n-m)$ is the number of triples with m as the middle entry. Therefore the generating function for the number of tableaux with distinct entries in the middle row is given by

$$\prod_{m=1}^{n-1} (1 + m(n-m)x) = \sum_{|\mathbf{p}|=n} \frac{n!}{s_{\mathbf{p}}}\, \frac{n!}{s_{\mathbf{p}'}}\, x^{n-\ell(\mathbf{p})}$$

This is closely related to the left hand side of (1.4). Indeed, rewriting (1.4) we have

$$F(x) = \sum_{|\mathbf{p}|=n} \left(\prod_i \frac{p_i}{2^{p_i - 1}} \right) \frac{n!}{s_{\mathbf{p}}}\, \frac{n!}{s_{\mathbf{p}'}}\, x^{n-\ell(\mathbf{p})}$$

The weights $\prod_i p_i/(2^{p_i-1})$ in the above right-hand side are less than one, and the desired conclusion follows.

The other identities derived in Section 3 have similar enumerative interpretations. The gist of these interpretations is that various measures that count sparseness of triples relative to their largest entries correspond to certain other measures that count sparseness relative to the middle entries. These developments will be reported on elsewhere.

5.2. **Integrable systems.** It has already been stated that spectral residues are meant to furnish a kind of spectral information for formal differential equations with a parameter. It is natural then to enquire for a physical interpretation of the spectral residue sequence. We will show that it is makes sense to regard the spectral residues as a formalized version of the potential's scattering data.

Such an interpretation comes about when we inquire about the evolution of the spectral residues concomitant to the evolution of the potential function with respect to the KdV equation.

Proposition 5.1. *Suppose that the potential series $U(z,t)$ evolves according to the non-commutative version of the KdV equation (see* [16] *for a list of references):*

$$U_t = D^3(U) - 3D(U)U - 3\,UD(U),$$

where $D = z\partial_z$. Then, the spectral residues ρ_n evolve according to the rule

$$\partial_t(\rho_n) = n^3\rho_n.$$

The proof is straightforward; it relies on the following identity

$$(p_1 + \ldots + p_n)^3 - (p_1^3 + \ldots + p_n^3) = \sum_j (p_1 + \ldots + p_j)(p_{j+1} + \ldots + p_n)(p_j + p_{j+1}). \quad (5.39)$$

Thus if we interpret the n in ρ_n as square roots of the energy (see the discussion following equation (2.11)) we see that the spectral residues evolve according to the same rule as the scattering data (reflection and normalization coefficients) of KdV wave profiles.

We now have the following quite natural interpretation of the transformations relating the coefficients of $U(z)$ and the spectral residues, i.e. equations (1.1) and (1.2): theses inversion formulas describes a formalized version of the well-known inverse scattering transformation for the KdV equation [17]. As such, the soliton solutions correspond to finitely supported residue sequences, and indeed it is possible to recover the well known n-soliton formula [18, 19] by means of equation (1.2).

It must be noted that the formula (1.2) and its relation to solitons has been established previously by Wadati and Sawada [19]. The identity (5.39) also occurs in that paper. Indeed, these authors went so far as to derive the Gelfand-Levitan-Marchenko equation from (1.2). This is not surprising, in light of the above interpretation of (1.2) as the restorative half of the inverse scattering transform. The missing idea in Wadati and Suwada's article was equation (1.1): the forward portion of the formalized I.S.T.

The next logical step in this development is the representation of other integrable systems by means of combinatorial inversions. It is reasonable to expect that the spectral residue formalism will be useful in translating the spectral problems that represent integrable equations into appropriate analogues of (1.1). The possible benefits of such a program are two-fold. First, the translation of scattering transforms for other integrable systems may yield new combinatorial results on par with the results for the Schrödinger equation reported on in this article. Second, by analogy with harmonic analysis it may be feasible to characterize classes of solutions to KdV by means of spectral residues. We already know that solitons correspond to finitely supported residue sequences. Is it possible to characterize compactly supported or exponentially decaying solutions in terms of an appropriately rapid decay

of the residue sequence? If such an approach is successful, it seems reasonable that it could be carried over to other integrable equations. These matters call for further investigation.

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